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Non-relativistic limit of the Einstein equation

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Abstract

In particular cases of stationary and stationary axially symmetric space-time passage to non-relativistic limit of Einstein equation is completed. For this end the notions of absolute space and absolute time are introduced due to stationarity of the space-time under consideration. In this construction absolute time is defined as a function t on the space-time such that ∂_t is exactly the Killing vector and the space at different moments is presented by the surfaces $t = \text{const}$. The space-time metric is expressed in terms of metric of the 3-space and two potentials one of which is exactly Newtonian gravitational potential Φ, another is vector potential $\vec{A}$ which, however, differs from vector potential known in classical electrodynamics. In the first-order approximation on Φ/c^2, $|\vec{A}|/c$ Einstein equation is reduced to a system for these functions in which left-hand sides contain Laplacian of the Newtonian potential, derivatives of the vector potential and curvature of the space and the right-hand sides do 3-dimensional stress tensor and densities of mass and energy. Subj-class: Classical Physics. Keywords: general relativity; non-relativistic limit; non-flat space.
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1 Introduction

Newtonian theory of gravitation was used as the underlying base when constructing general relativity and remains a good non-relativistic approximation to it. However, prediction of the Lense-Thirring effect few years after it was created gave rise to the idea of Coriolis or gravimagnetic interaction together with the corresponding field which requires some extension of the Newtonian theory. Discussing a question which could well be answered centuries ago would show that the Newtonian theory is incomplete. Indeed, consider a spherical mass of self-gravitating matter in equilibrium without rotation. The same equilibrium state should exist in rotating frame. However, passage to a frame rotating with angular velocity ω produces in cylindric coordinates the centrifugal potential $\omega^2\rho^2/2$ while rotation of the matter in this frame produces exactly the same centrifugal potential. Instead of canceling each other, these two add that breaks the equilibrium. Therefore equilibrium of non-rotating matter in a rotating frame cannot be described in Newtonian theory. However, *The laws of physics must be of such a nature that they apply to systems of reference in any kind of motion* [1]. Inclusion of gravimagnetic field whose strength in this example equals $H_z = \omega^2\rho$ improves the situation because velocity of matter multiplied by this strength cancels both the centrifugal potentials.

More recent development of astrophysics brought new questions to be answered. One of them consists in the following. Infinitesimally-thin disk is an important astrophysical model.

Its mass density is singular in space that can be presented as one-dimensional δ-function multiplied by surface density. Due to Einstein equation curvature of the space-time has the same singularity, hence, the factor c^{-2} where c is speed of light, which the curvature is multiplied by, does not provide flatness assumed in the Newtonian theory. In other words, passage to "Newtonian" limit does not lead to Newtonian theory. This fact signifies an apparent need for the next post-Newtonian approximation in which c^{-2} terms are included and the space is not assumed to be flat.

Another reason to build this approximation is need for models of neutron stars in astrophysics. Neutron stars are known to possess the highest possible mass density in the nature, which is about $10^{14} g \cdot cm^{-3}$. At the same time velocity of matter in these objects does not exceed 0.1 c, therefore, non-relativistic mechanics provides sufficient accuracy if applied under these conditions. Gravitational potential measured in units of c^2 does not exceed the value of 10^{-2} [2], thus, is usable. Then equilibrium of matter might well be described in terms of Newtonian or post-Newtonian approximation which assumes that time is absolute, space is flat and absolute and gravitational field is presented by two potentials: one scalar (gravistatic) and one vector (gravimagnetic, which produces the Coriolis force [3, 4]). However, the absolute space cannot be put flat as in these approximations. The mass density multiplied by gravitational constant which is $1.87 \cdot 10{-}27\ cm\ g^{-1}$ yields typical value of curvature $3.9 \cdot 10^{-13} cm^{-2}$. Though this value seems to be negligibly small, employing typical radius of neutron star which is about $10^6 cm$ as the unit of length leads to spatial curvature about 0.39 that cannot be neglected.

Consequently, the most appropriate approximation to general relativity to be used for this end would be a non-relativistic theory with absolute space and time and gravitational field specified by the potentials and geometry of the space. Generally accepted equations of this theory are

$$\Box h^{\alpha\beta} = -16\pi\tau^{\alpha\beta},$$

where $\Box \equiv \partial^2/\partial t^2 + \nabla^2$ is the 'flat-space-time wave operator', $h^{\alpha\beta}$ is a "gravitational tensor potential" related to the deviation of the space-time metric from its Minkowski form by the formula $h^{\alpha\beta} \equiv \eta^{\alpha\beta} - (-g)^{-1/2} g^{\alpha\beta}$, g is the determinant of $g_{\alpha\beta}$ [5]. In this work another approximation to Einstein equation is considered, in which all velocities are assumed to be negligibly small compared with c, therefore time and space are absolute, but the space is not assumed to be flat. The goal of the present work is to derive exact consequences from the Einstein equation under these assumptions.

In order to determine what will be called "space" and "time" we introduce a coordinate system $\{t, x^i\}$ in which the coordinate t plays the key role because thereafter this coordinate will be used as absolute time and surfaces $t = const$ – absolute space. As a function on the space-time this coordinate is to be chosen such a way that square of its gradient $\langle dt\,, dt\rangle$ is close enough to unit due to the approximation chosen. Afterwards, whatever coordinate transformations we make, the coordinate t remains unchanged. It must be noted that t is only a coordinate, not proper time of an observer because length of its gradient dt is not exactly one.

By spatial metric we mean Riemannian metric of the surfaces $t = const$

$$\left\langle dx^i\,, dx^j \right\rangle |_{t=const} \equiv g^{ij}.$$

Since functions x^i on the space-time are quite arbitrary, genuine gradients dx^i are not orthogonal to dt: $\langle dt\,, dx^i\rangle \;\equiv\; A^i/c$ and genuine scalar products $\langle dx^i\,, dx^j\rangle$ differ from g^{ij}, however the difference is of order c^{-2} and we neglect it. By the result we have space-time metric in the form

$$\langle dt\,, dt\rangle \;=1-\frac{2\Phi}{c^2},\; \left\langle dt\,, dx^i\right\rangle \;=\frac{A^i}{c},\; \left\langle dx^i\,, dx^j\right\rangle \;=-g^{ij}-\frac{A^iA^j}{c^2}. \tag{1}$$

It will be shown below that the term Φ/c^2 in g^{tt} cannot be neglected, as well as $g^{t\varphi}$ which is of order c^{-1} while for purely spatial components they are to be omitted.

2 Coordinate transformations

Intersection of three coordinate surfaces $x^i = const$ is coordinate line, a time-like curve which specifies the time axis through given point of the space. In general, the time axis and the space are not orthogonal because, on one hand, this curve is specified only by coordinate surfaces $x^i = const$ and, on the other hand, these surfaces are introduced regardless of choice of the surfaces $t = const$. Non-orthogonality of space and time reveals in the components g^{ti} of the metric (1). In this section we consider coordinate transformations under which the space and time remain unchanged.

Invariance of the space means that the transformations do not touch the coordinate t, thus, they are of the form

$$t \to t,\;\; x^i \to y^a(x^i, t).$$

These transformations are of two different kinds. Ordinarily coordinate transformations

$$t \to t,\;\; x^i \to y^a(x^i)$$

leave the time axis immobile everywhere. Another kind is given by changes of frame of reference. Its simplified form is

$$x^i \to y^i = x^i + f^i(t) \tag{2}$$

and the general form is superposition of transformations of both kinds. Hereafter we employ only simplified form (2) of this transformation.

Differentiation of the equation (2) gives

$$dy^i = dx^i + f'^i dt.$$

It is seen that the metric (1) remains invariant under change of frame, proviso that coefficients A^i transform as

$$A^i \to A^i + cf'^i\left(1-\frac{2\Phi}{c^2}\right). \tag{3}$$

3 Hamilton-Jacobi equation

Hamilton-Jacobi equation for time-like geodesics has the form

$$\left(1-\frac{2\Phi}{c^2}\right)\left(\frac{\partial S}{\partial t}\right)^2+\frac{2A^i}{c}\,\frac{\partial S}{\partial t}\,\frac{\partial S}{\partial x^i}-\langle \nabla S\,, \nabla S\rangle \;=m^2c^2 \tag{4}$$

where ∇S stands for the spatial part of the 1-form dS. Assume that there exists a function W satisfying the equations

$$2m\frac{\partial W}{\partial t} = \left(\frac{\partial S}{\partial t}\right)^2 - m^2c^2,\ \ \nabla S = \nabla W \tag{5}$$

(for example, S_t depends only on t). Then the equation (4) can be rewritten as follows:

$$\left(1-\frac{2\Phi}{c^2}\right)\left(m^2c^2+2m\frac{\partial W}{\partial t}\right)+\frac{2A^i}{c}\sqrt{m^2c^2+2m\frac{\partial W}{\partial t}}-\\ -\langle\nabla W\,,\nabla W\rangle\ = m^2c^2.$$

Removing the parentheses and omitting negligibly small terms yields:

$$\frac{1}{2m}\langle\nabla W\,,\nabla W\rangle\ - A^i\frac{\partial W}{\partial x^i}+m\Phi = \frac{\partial W}{\partial t} \tag{6}$$

that almost coincides with well-known form of of non-relativistic Hamilton-Jacobi equation for a particle moving in gravitational potential Φ and vector potential $\vec{A}$. The only difference is that this equation does not contain the term proportional to $|\vec{A}|^2$ which usually presents in the standard version where 1-form $A \equiv A_i dx^i$ enters only in combination $\nabla W - A$. The origin of this difference will be discussed later.

Reduction of the Hamilton-Jacobi equation for isotropic geodesics is similar and leads to that for geodesics on the 3-space with metric g^{ij}. Indeed, in this case $m = 0$ and all terms containing negative powers of c can be neglected. Consequently, world lines of massless particles are just geodesics of the space which they should be in a non-relativistic theory. As for strictly space-like geodesics ($m^2 < 0$), they have no physical meaning in the case.

Usually, when deriving Hamilton-Jacobi equation one starts with Lagrangian, then introduces generalized momenta, obtains the form of Hamiltonian and substitutes ∇S for the momentum. Vector potential can be included into this scheme two different ways. By the result, the final form of Hamilton-Jacobi equation depends on the way it is done. Consider two examples.

Let $\vec{u}$ be a field of velocities which specifies motion of a frame w.r.t. some rest or inertial one. Let a mass point move in this frame without any force acting on it. Then its Lagrangian $L(\vec{v})$ is

$$L(\vec{v}) = \frac{m}{2}(\vec{v}-\vec{u})^2 = \frac{m}{2}\vec{v}^2 - m\vec{u}\cdot\vec{v}+\frac{m}{2}\vec{u}^2$$

where the vector $\vec{u}$ enters as a vector potential and its square does as scalar potential.

By definition, generalized momentum $\frac{\partial L}{\partial \vec{v}}$ is

$$\vec{p} = m(\vec{v}-\vec{u}),$$

thus,

$$H = \vec{p}\cdot\vec{v} - L = \vec{p}\cdot\left(\frac{\vec{p}}{m}+\vec{u}\right) - \frac{\vec{p}^2}{m} = \frac{\vec{p}^2}{2m}+\vec{p}\cdot\vec{u},$$

so, square of the vector potential does not appear in the Hamiltonian and finally, the Hamilton-Jacobi equation does not contain it. The following example shows that usually it is not so.

Lagrangian of a particle with unit charge in a vector potential of magnetostatic field $\vec{A}$ has the form

$$L = \frac{m\vec{v}^2}{2} - \vec{A} \cdot \vec{v}.$$

By definition, generalized momentum is

$$\vec{p} = m\vec{v} - \vec{A}$$

that coincides with the previous case. However, Hamiltonian is different:

$$H = \frac{1}{m}\vec{p} \cdot (\vec{p} + \vec{A}) - L = \frac{1}{2m}(\vec{p}^2 + 2\vec{A} \cdot \vec{p} + \vec{A}^2).$$

This expression contains square of vector potential that is very inconvenient when constructing analytical solutions. Thus, if vector potential specifies motion of the frame chosen, its square appears in Lagrangian but does not appear in the Hamiltonian and Hamilton-Jacobi equation whereas if it specifies a kind of connection its square does not appear in Lagrangian but does in Hamiltonian and Hamilton-Jacobi equation. Therefore hereafter we distinguish the two kinds of vector potentials and call that of gravimagnetic field "Coriolis potential" as it was called from the very beginning by Einstein.

The following remark must be made about rotating frames. If the space and gravitational field possess axial symmetry it is convenient to use a coordinate system with one of coordinates being azimuthal angle φ such that ∂_φ is the Killing vector. Then passage to rotating frame is (3) that produces Coriolis potential

$$A^\varphi \to A^\varphi + c\omega\left(1 + \frac{2\Phi}{c^2}\right) \tag{7}$$

.

4 Orthonormal frames and connection

When reducing Einstein equation we shall employ orthonormal vector $\{\vec{n}_a\}$ and co-vector $\{\nu^a\}$ frames and E. Cartan structure equations

$$\begin{aligned} d\nu^a + \omega_b{}^a \wedge \nu^b = 0 \\ \Omega_a{}^b = d\omega_a{}^b + \omega_c{}^b \wedge \omega_a{}^c. \end{aligned} \tag{8}$$

It is convenient to introduce orthonormal frames adapted to the surfaces $t = const$:

$$\begin{aligned} \nu^0 = \left(1 + \frac{\Phi}{c^2}\right) dt,\ \nu^a = h_i^a dx^i - \frac{A^i}{c} dt\, A^a \equiv h_i^a A^i \\ \vec{n}_0 = \left(1 - \frac{\Phi}{c^2}\right) \frac{\partial}{\partial t} - \frac{A^i}{c}\frac{\partial}{\partial x^i},\ \vec{n}_a = h^i_a \frac{\partial}{\partial x^i}, \end{aligned} \tag{9}$$

where h^i_a, h_i^a are matrices obeying the following conditions:

$$\delta^{ab} h^i_a h^j_b = g^{ij},\ h^i_a h_i^b = \delta^b_a.$$

By construction, the triplet of $\{\nu^a\}$, $a \neq 0$ constitutes an orthonormal frame for surfaces $t = const$ because they are orthogonal to dt and triplet of the vectors $\{\vec{n}_a\}$ – an orthonormal frame for the surfaces because $\vec{n}_a$ are tangent to them. Since the frames are adapted to the absolute space as it was defined above and the space is unaffected by change of frame of reference the frames (2) are invariant with respect to transformations (3).

Now our task is to obtain the connection 1-form for the frames from the first structure equation (8). Before doing this we introduce the following triplet of auxiliary 1-forms:

$$\theta^a \equiv h^a_i dx^i = \nu^a + \frac{A^a}{c} dt$$

which are not purely spatial. Exterior derivatives of the 1-forms ν^0 and ν^a are:

$$d\nu^0 = -\frac{1}{c^2} dt \wedge d\Phi \approx -\frac{1}{c^2} \nu^0 \wedge d\Phi$$
$$d\nu^a = d\theta^a + \frac{1}{c} dt \wedge dA^a = -\psi_b{}^a \wedge \theta^b + \frac{1}{c} dt \wedge dA^a$$

where we have introduced the connection 1-form ψ^a_b assuming it to satisfy the equation

$$d\theta^a + \psi_b{}^a \wedge \theta^b = 0.$$

Thus, $\psi_b{}^a$ is the purely spatial part of the connection. Indeed,

$$d\nu^a = -\psi_b{}^a \wedge \left(\nu^b + \frac{A^b}{c} dt\right) + \frac{1}{c} dt \wedge dA^a =$$
$$= -\psi_b{}^a \wedge \nu^b + \frac{1}{c} dt \wedge \left(dA^a + \psi_b{}^a A^b\right) = -\psi_b{}^a \wedge \nu^b + \frac{1}{c} dt \wedge DA^a$$

where DA^a stands for covariant exterior derivative of the 0-form A^a. Finally,

$$d\nu^a = -\psi_b{}^a \wedge \nu^b + \frac{1}{c} \nu^0 \wedge DA^a$$

where terms of order c^{-3} have been ignored.

Solution of the first structure equation for the space-time is

$$\omega_b{}^0 = \frac{1}{c^2} (\vec{n}_b \circ \Phi) \nu^0 - \frac{1}{2c} (D_b A_a + D_a A_b) \nu^a$$
$$\omega_a{}^b = \psi_a{}^b + \frac{1}{2c} (D_c A_a - D_a A_c) \delta^{bc} \nu^0$$

where $\vec{n}_b \circ \Phi$ is action of the differential operator $\vec{n}_b$ (9) on the function Φ. One more difference between Coriolis potential A appeared here and genuine vector potential reveals in these equations: genuine vector potential never appears in symmetrized derivatives like the last term in the first line. Hereafter we pass from general case to the case of stationary axially-symmetric space-time.

5 Stationary axially-symmetric field

Let $\{u, v, \varphi\}$ be a coordinate system for axially-symmetric space in which ∂_φ is the Killing vector. All functions of the field depend only on the coordinates u and v. The adapted frames (9) are

$$\nu^0 = \left(1 + \frac{\Phi}{c^2}\right) dt, \quad \nu^1 = \frac{du}{\sigma}, \quad \nu^2 = \frac{dv}{\sigma}, \quad \nu^3 = \rho\left(d\varphi - \frac{A}{c}dt\right)$$
$$\vec{n}_0 = \left(1 - \frac{\Phi}{c^2}\right)\frac{\partial}{\partial t} - \frac{A}{c}\frac{\partial}{\partial\varphi}, \quad \vec{n}_1 = \sigma\frac{\partial}{\partial u} \quad \vec{n}_2 = \sigma\frac{\partial}{\partial v} \quad \vec{n}_3 = \frac{1}{\rho}\frac{\partial}{\partial\varphi}.$$

Note that in our approximation scheme we can write

$$d\varphi = \frac{\nu^3}{\rho} + \frac{A}{c}\nu^0.$$

It is easy to obtain the purely spatial part of connection:

$$\psi_1{}^2 = \sigma_v\nu^1 - \sigma_u\nu^2 = -\psi_2{}^1 \quad \psi_2{}^3 = \frac{\rho_v}{\rho}\sigma\nu^3 = -\psi_3{}^2, \quad \psi_3{}^1 = -\frac{\rho_u}{\rho}\sigma\nu^3 = -\psi_1{}^3.$$

Below we use denoting

$$H_{a3} = \vec{n}_a \circ (\rho A)$$

and keep in mind that lifting a spatial index changes the sign. Exterior derivatives of the 1-forms ν^a are

$$\begin{aligned}
d\nu^0 &= -\frac{\vec{n}_1\circ\Phi}{c^2}\nu^0\wedge\nu^1 - \frac{\vec{n}_2\circ\Phi}{c^2}\nu^0\wedge\nu^2, \quad d\nu^1 = -\psi_2{}^1\wedge\nu^2, \quad d\nu^2 = -\psi_1{}^2\wedge\nu^1,\\
d\nu^3 &= \frac{H_1{}^3}{c}\nu^0\wedge\nu^1 + \frac{H_2{}^3}{c}\nu^0\wedge\nu^2 - \psi_1{}^3\wedge\nu^1 - \psi_2{}^3\wedge\nu^2.
\end{aligned}$$

The corresponding form of connection is:

$$\begin{aligned}
\omega_0{}^1 &= \frac{\vec{n}_1\circ\Phi}{c^2}\nu^0 + \frac{H^1{}_3}{2c}\nu^3 = \omega_1{}^0, \qquad \omega_1{}^2 = \psi_1{}^2 = -\omega_2{}^1\\
\omega_0{}^2 &= \frac{\vec{n}_2\circ\Phi}{c^2}\nu^0 + \frac{H^2{}_3}{2c}\nu^3 = \omega_2{}^0, \qquad \omega_2{}^3 = \psi_2{}^3 - \frac{H_2{}^3}{2c}\nu^0 = -\omega_3{}^2\\
\omega_0{}^3 &= \frac{H^1{}_3}{2c}\nu^1 + \frac{H^2{}_3}{2c}\nu^2 = \omega_3{}^0, \qquad \omega_3{}^1 = \psi_3{}^1 - \frac{H^1{}_3}{2c}\nu^0 = -\omega_3{}^2.
\end{aligned}$$

Our next task is to obtain the curvature 2-form of the space-time that will be done with use of the structure equations (8).

6 Curvature of stationary axially-symmetric space-time

Collecting similar terms in the r.h.s. of the second structure equation (8) where exterior derivatives and exterior products of components of the connection 1-form are substituted, we obtain

the curvature 2-form and the following non-zero components of the Riemann tensor which contribute the Einstein equation:

$$
\begin{aligned}
R_0{}^1{}_{01} &= -2\frac{D_1(\vec{n}_1\circ\Phi)}{c^2} + \frac{H_1{}^3H^1{}_3}{c^2}, \qquad R_0{}^1{}_{02} = -2\frac{D_1(\vec{n}_2\circ\Phi)}{c^2} + \frac{H^1{}_3H_{23}}{c^2},\\
R_0{}^1{}_{31} &= \frac{1}{c}(D_1H^1{}_3 + D_3H^3{}_3 + D_3H^1{}_1), \qquad R_0{}^2{}_{02} = -2\frac{D_2(\vec{n}_2\circ\Phi)}{c^2} + \frac{H_2{}^3H^2{}_3}{c^2},\\
R_0{}^2{}_{23} &= -\frac{1}{c}(D_2H^2{}_3 + D_3H^3{}_3 + D_3H^2{}_2), \qquad R_0{}^3{}_{03} = -2\frac{D_3(\vec{n}_3\circ\Phi)}{c^2} - \frac{H^2}{2c^2},\\
R_1{}^2{}_{12} &= K_1{}^2{}_{12} \qquad R_2{}^3 23 = K_2{}^3{}_{23} - \frac{H_2{}^3H_{23}}{2c^2},\\
R_2{}^3{}_{31} &= K_2{}^3{}_{31} + \frac{H_1{}^3H_{23}}{2c^2},\ R_3{}^1{}_{31} = K_3{}^1{}_{31} - \frac{H_1{}^3H_{13}}{2c^2}.
\end{aligned}
\tag{10}
$$

; $K_a{}^b{}_{cd}$ denotes Riemann tensor of the space:

$$
\frac{1}{2}K_a{}^b{}_{cd}\nu^c\wedge\nu^d \equiv d\psi_a{}^b + \psi_c{}^b\wedge\psi_a{}^c. \tag{11}
$$

Our next task is to reduce the Einstein equation to its three-dimensional form.

7 Ricci and Einstein tensors

Converting the Riemann tensor (10) yields the following components of the Ricci tensor and scalar curvature:

$$
\begin{aligned}
R_{00} &= -2\frac{\Delta\Phi}{c^2} + \frac{H^2}{2c^2}, \qquad R_{03} = \frac{D_aH^a{}_3}{c}\\
R_{11} &= -2\frac{D_1(\vec{n}_1\circ\Phi)}{c^2} + \frac{H_1{}^3H_{13}}{2c^2} + K_{11}, \quad R_{12} = -2\frac{D_1(\vec{n}_2\circ\Phi)}{c^2} - \frac{H_1{}^3H_{23}}{2c^2} + K_{12},\\
R_{22} &= -2\frac{D_2(\vec{n}_2\circ\Phi)}{c^2} + \frac{H_2{}^3H_{23}}{2c^2} + K_{22}, \qquad R_{33} = -2\frac{D_3(\vec{n}_3\circ\Phi)}{c^2} + \frac{H^2}{2c^2} + K_{33}
\end{aligned}
$$

where $\Delta\Phi$ is commonplace three-dimensional Laplacian

$$
\Delta\Phi = \delta^{ab}D_a(\vec{n}_b\circ\Phi)
$$

and we put $D_3H^a{}_a = 0$ because trace of H_{ab} is zero; K_{ab} and K – Ricci tensor of the space:

$$
K_{ab} = K_a{}^c{}_{bc}
$$

and its trace. Scalar curvature of the space-time is

$$
R = -\frac{4}{c^2} - K.
$$

As expected, this scalar does not depend on ${H_a}^b$ and, hence, on choice of the reference frame. Now we can compose the Einstein tensor. Its components are:

$$
\begin{aligned}
G_{00} &= \frac{1}{2}\left(K + \frac{H^2}{c^2}\right), \ G_{03} = \frac{1}{c} D_a {H^a}_3, \\
G_{11} &= \frac{2}{c^2}[D_1(\vec{n}_1 \circ \Phi) - \Delta\Phi] + \Gamma_{11} + \frac{{H_1}^3 H_{13}}{2c^2} \\
G_{12} &= \frac{2}{c^2} D_1(\vec{n}_2 \circ \Phi) + \Gamma_{12} - \frac{{H_1}^3 H_{23}}{2c^2} \\
G_{22} &= \frac{2}{c^2}[D_2(\vec{n}_2 \circ \Phi) - \Delta\Phi] + \Gamma_{22} + \frac{{H_2}^3 H_{23}}{2c^2} \\
G_{33} &= \frac{2}{c^2}[D_3(\vec{n}_3 \circ \Phi) - \Delta\Phi] + \Gamma_{33} + \frac{H^2}{c^2}
\end{aligned}
$$

where we have introduced Einstein tensor for the space:

$$
\Gamma_{ab} = K_{ab} - \frac{1}{2}\delta_{ab} K.
$$

Finally, Einstein equation reduces to the following system:

$$
\begin{aligned}
\frac{1}{2}\left(K + \frac{H^2}{c^2}\right) = \kappa T_{00} = \kappa\varepsilon, \qquad \frac{1}{c} D_a {H^a}_3 = \kappa T_{03} = \kappa J_3 \\
\frac{2}{c^2}[D_1(\vec{n}_1 \circ \Phi) - \Delta\Phi] + \Gamma_{11} + \frac{{H_1}^3 H_{13}}{2c^2} = \kappa T_{11} \\
\frac{2}{c^2} D_1(\vec{n}_2 \circ \Phi) + \Gamma_{12} - \frac{{H_1}^3 H_{23}}{2c^2} = \kappa T_{12} \\
\frac{2}{c^2}[D_2(\vec{n}_2 \circ \Phi) - \Delta\Phi] + \Gamma_{22} + \frac{{H_2}^3 H_{23}}{2c^2} = \kappa T_{22} \\
\frac{2}{c^2}[D_3(\vec{n}_3 \circ \Phi) - \Delta\Phi] + \Gamma_{33} + \frac{H^2}{c^2} = \kappa T_{33}
\end{aligned}
$$

where T_{ab} stands for stress-energy tensor of the matter; its components ε and J_3 are density of energy (including rest energy ρc^2) and the 3-component of the mass current. Remarkably, one of these equations reads

$$
\Delta\Phi + \frac{1}{4} K = 4\pi k \mu, \ k = \frac{16\pi}{c^2}
$$

where k is Newtonian gravitational constant, μ – mass density.

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